MATH2048 24-25 Midterm 1 Solution

1 (a). Pick $f_1, f_2 \in P_n(\mathbb{R})$, $c \in \mathbb{R}$,

Then $T(f_1 + cf_2) = (f_1 + cf_2) + (f_1 + cf_2)'$

$= f_1 + f_1' + c(f_2 + f_2')$

$= T(f_1) + cT(f_2)$

$\therefore$ T is linear. $(T(0) = 0)$

Pick $f \in N(T) \Rightarrow f - f' = 0$

$\Rightarrow f = f'$

Note that $\deg f > \deg f'$ except $f = 0$.

$\therefore N(T) = \{0\} \Rightarrow T$ one-to-one.

Since T is a linear operator on fin-dim VS,

T one-to-one $\Rightarrow$ T onto.

$\therefore$ T is an isomorphism

(b). $[f]_\beta = (4, -2, 1)^T$

$$[I]_\beta^\gamma = ([I]_\gamma^\beta)^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$\therefore [f]_\gamma = [I]_\beta^\gamma [f]_\beta = (6, -3, 1)^T$ //

(c). $[T]_\beta [f]_\beta = [T(f)]_\beta \neq [T(f)]_\gamma = [T]_\gamma [f]_\gamma$

2(a).
$$A = \begin{pmatrix} 3 & 3 & 9 \\ -6 & 3 & 0 \\ -1 & -5/2 & -6 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 1 \\ & 1 & 2 \\ & & \end{pmatrix}$$

$\Rightarrow N(L_A) = \text{span}\{(1, 2, -1)\}$.

$$A^T = \begin{pmatrix} 3 & -6 & -1 \\ 3 & 3 & -5/2 \\ 9 & 0 & -6 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & -2/3 \\ & 1 & -1/6 \\ & & \end{pmatrix}$$

$\Rightarrow R(L_A) = \text{span}\{(1, 0, -2/3), (0, 1, -1/6)\}$.

Check that

$(1, 2, -1) = (1, 0, -2/3) + 2(0, 1, -1/6)$

$\therefore N(L_A) \cap R(L_A) \neq \{0\} \Rightarrow$ False.

(b). View $M_{3\times3}(\mathbb{R})$ as $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$

Then $N(\tilde{L}_A) = \text{span}\{(\vec{v}^T, 0, 0), (\vec{0}, \vec{v}^T, \vec{0}), (\vec{0}, \vec{0}, \vec{v}^T)\}$

where $\vec{v} = (1, 2, -1) \in N(L_A)$

$\therefore \text{nullity}(\tilde{L}_A) = 3$ and $\text{rank}(\tilde{L}_A) = 6$

3. $(\Rightarrow)$ Suppose $\dim \ker T_1 = \dim \ker T_2$.

Then $\exists$ $\beta_1 = \{v_1^i\}$ and $\beta_2 = \{v_2^i\}$ bases of V

s.t. $T_1(v_1^j) = T_2(v_2^j) = 0$ $\forall j \leq k$.

Note that $\operatorname{rank} T_1 = \operatorname{rank} T_2$ by Rank-Null. Thm

Then $\exists$ $\gamma_1 = \{w_1^i\}$ and $\gamma_2 = \{w_2^i\}$ bases of W

s.t. $T_1(v_1^j) = w_1^j$ $\forall j > k$.

$T_2(v_2^j) = w_2^j$ $\forall j > k$

Construct $R: V \to V$ s.t. $R(v_1^i) = v_2^i$ $\forall i$

$S: W \to W$ s.t. $S(w_2^i) = w_1^i$ $\forall i$

Then $\forall j \leq k$: $ST_2R(v_1^j) = ST_2(v_2^j) = 0 = T_1(v_1^j)$

$\forall j > k$: $ST_2R(v_1^j) = ST_2(v_2^j) = S(w_2^j)$

$= w_1^j = T_1(v_1^j)$

$\therefore$ $T_1 = ST_2R$ as all basis vectors agree,

and S, R invertible as they maps basis to basis $\Rightarrow$ full rank.

$(\Leftarrow)$ Suppose $T_1 = ST_2R$ (S, R invertible)

Then $x \in N(T_1) \Leftrightarrow ST_2R(x) = 0$

$\Leftrightarrow T_2R(x) = 0$ as S full-rank

$\Leftrightarrow x \in R^{-1}(N(T_2))$

R full rank $\Rightarrow \dim N(T_2) = \dim R^{-1}(N(T_2)) = \dim N(T_1)$

4. Let $\mathcal{F} = \{ S \subset F , S$ L.I. and $\operatorname{span}(S) \cap W = \{0\} \}$

Write $S_1 \leq S_2$ if $S_1 \subseteq S_2$.

For every chain $\{S_i\} \subset \mathcal{F}$, we claim that (1) $\bigcup S_i$ is an upper bound of $\{S_i\}$ and (2). $\bigcup S_i \in \mathcal{F}$.

(1) is obvious as $S_j \subseteq \bigcup S_i \quad \forall j$.

(2). Pick any $S \subset \bigcup S_i$, then $\exists\, S_j \in \{S_i\}$ s.t. $S \subset S_j \Rightarrow S$ L.I.

Pick $x \in \operatorname{span}(\bigcup S_i) \Rightarrow x = \sum_{k=1}^{n} a_i x_i$, then $\exists\, S_j$ s.t. $x_i \subset S_j \quad \forall i$.

$\therefore$ By Zorn's Lemma, $\exists$ maximal element β in $\mathcal{F}$. We want to show $V = \operatorname{span}(\beta) \oplus W$.

$\operatorname{Span}(\beta) \cap W = \{0\}$ as $\beta \in \mathcal{F}$

If $\operatorname{span}(\beta) + W \subsetneq V$, pick $x \in V \setminus \operatorname{span}(\beta) + W$.

Then $\operatorname{span}(\beta \cup \{x\}) \cap W = \{0\}$ and $\beta \cup \{x\}$ L.I. Contradiction.

$\therefore V = \operatorname{span}(\beta) \oplus W$

5 (a). Note that $\vec{0} \in W_n$

Pick $w_1 = (x_1^1, x_2^1, \ldots)$, $w_2 = (x_1^2, x_2^2, \ldots) \in W_n$,

$c \in \mathbb{R}$

Then $\sum_{k=1}^{n} k^2 (x_k^1 + c x_k^2) = \sum_{k=1}^{n} k^2 x_k^1 + c \sum_{k=1}^{n} k^2 x_k^2 = 0$

$\therefore$ $w_1 + c w_2 \in W_n$. $\Rightarrow$ W_n is a subspace.

Write $W = \bigcup_{n=1}^{\infty} W_n$

Note that $\vec{0} \in W$

Pick $w \in \bigcup W_n$ $\Rightarrow$ $w \in W_n$ for some n

$\Rightarrow$ $cw \in W_n$ $\forall c \in \mathbb{R}$

$\Rightarrow$ $cw \in W$

If $w \in W_i$, then $\sum_{k=0}^{j} k^2 x_k = \sum_{k=1}^{i} k^2 x_k = 0$ $\forall j \geq i$

as $x_k = 0$ $\forall k > i$

$\therefore$ $W_i \subseteq W_j$ $\forall i \leq j$.

So pick $w_1, w_2 \in W$, let

$w_1 \in W_i$ and $w_2 \in W_j$, WLOG assume $i \leq j$

Then $w_1 \in W_j$ $\Rightarrow$ $w_1 + w_2 \in W_j \subset W$

$\therefore$ W is a subspace.

(b). Pick any $w = (x_1, \dots) \in V$, let $c = \sum_{k=1}^{\infty} k^2 x_k$

Then $w = -ce_1 + (-ce_1 + w)$

where $-ce_1 + w \in W$.

Check that $\text{span}(\{e_1\}) \cap W = \{0\}$

$\therefore \text{span}(\{e_1\}) \oplus W = V$

$\Rightarrow V/W = \{ce_1 + W : c \in \mathbb{R}\}$.

$\Rightarrow \dim(V/W) = 1$.

(c). Let $U : W \to V$

$(x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$

$T : V \to W$

$(x_1, x_2, \dots) \mapsto (c, x_1, x_2, \dots)$

where $c = -\sum_{k=1}^{\infty} (k+1)^2 x_k$

Check that U, T well-defined,

$UT = I_V$ and $TU = I_W$

$\therefore U = T^{-1} \Rightarrow V \cong W$.